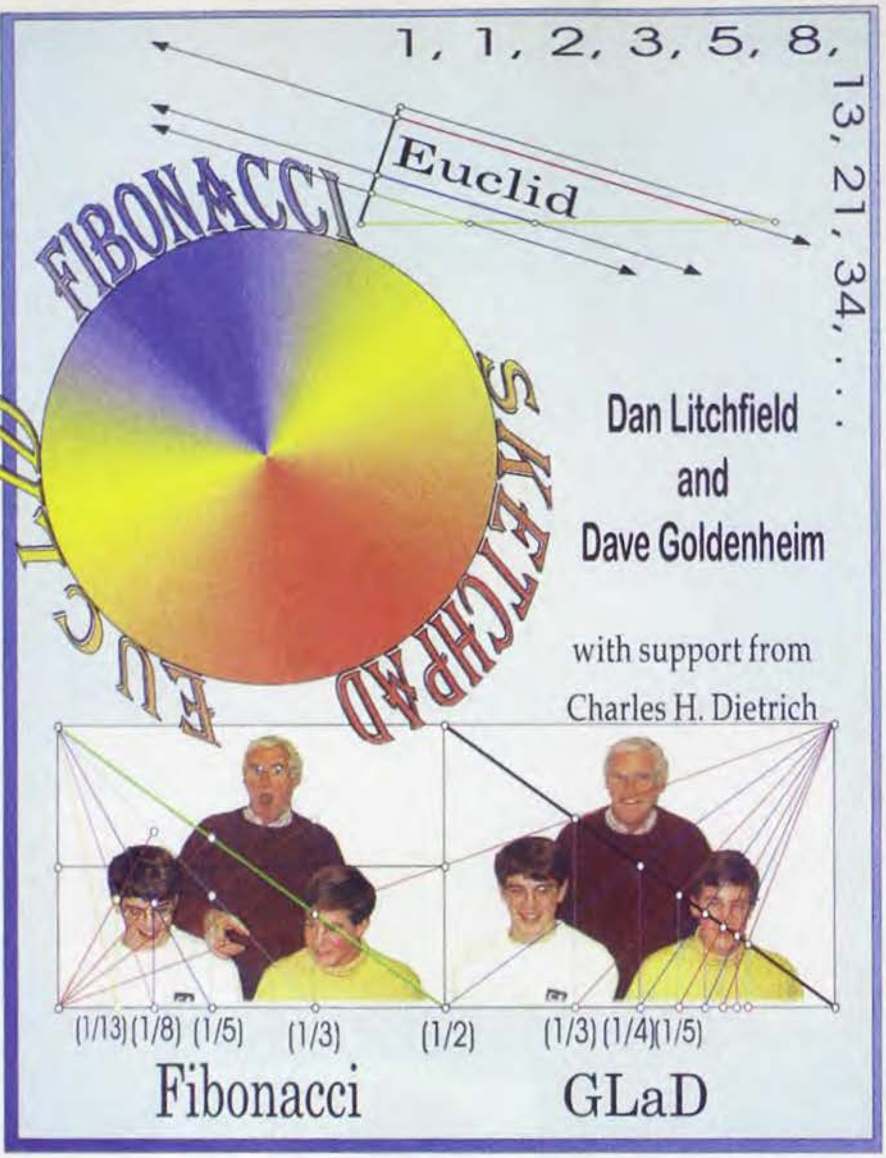

In late June 1995 I began teaching geometry and trigonometry to David Goldenheim and Daniel Litchfield at Greens Farms Academy's summer session. Their reputations in mathematics had reached my ears long before I worked with them, so I was prepared for a fascinating and challenging time. I had, however, no idea of the adventure on which we were to embark.

In mid-July I asked them to solve the following problem: Divide any line segment into a regular partition of any number of parts. "And by the way, fellas," I said, "You don't have a prayer of figuring this out." This exercise is found in most geometry textbooks in the form of a construction and is itself a variation of Proposition 10, Book 6 of Euclid's *Elements:* "to cut a given uncut straight line similarly to a given cut straight line."

Within two hours, they announced that they had solved it, not with compass and straightedge but on a computer by using The Geometer's Sketchpad. Was that approach okay? they asked. I, of course, said, "Sure, sure." But when I saw it, I nearly had a heart attack. My instincts after thirty-three years of teaching mathematics screamed at me that this construction was unique and original—in fact, a discovery. Perhaps it was only the second such construction of the problem since Euclid's time, maybe even since antiquity. My mind raced on and on.

They then showed me an additional construction. This one yielded a pattern that every spirited young mathematician would recognize—as they had—and it was nothing less than the Fibonacci sequence. The Fibonacci sequence? I had to lie down.

When I rallied, I immediately got them down to some serious mathematics. First, we proved the first construction, synthetically and analytically. Finally, we proved it both ways using the principle of mathematical induction. As for the second construction, its proof was considerably harder, but we were able, at the end of the day, to prove it analytically.

And, oh yes, I shall be eternally grateful to those two amazing ninth graders. Boys, take over!

THE CHALLENGE:

Divide any line segment into a regular partition of any number of parts.

THE GLaD CONSTRUCTION (DAN)

We used a computer program called The Geometer's Sketchpad to experiment. From any line segment we constructed any rectangle because a rectangle offered more flexibility than a square and more points than a triangle. Since the diagonals of any rectangle intersect at a point that lies on the perpendicular bisector of any side of the rectangle, we started to experiment by connecting points to form various segments in the hopes that one of the points of intersection of two segments would lie directly above a point of trisection of the base. Twenty minutes later, by experimenting, by applying trial and error, and by using the computer to measure distances, we found a point that did just that. This point was the keystone of our discoveries. After the discovery of the "one-third" point, and after our flawed attempts at finding the pattern we were sure was there, Dave discovered the "one-seventh" point. Then, sooner rather than later, our pattern suddenly jumped right out at us and became obvious **(fig. 1).**

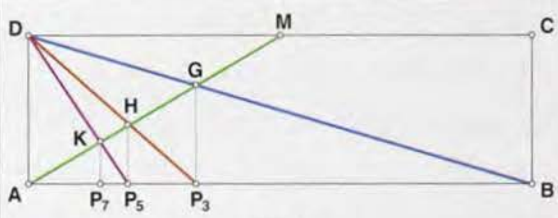

Fig. 1
Is $AP_n = (1/n)\,AB$?

Our method of construction follows:

- On any line segment **AB** construct any rectangle **ABCD.**
- Find midpoint **M** of **CD,** where $DM = (1/2)DC = (1/2)AB$.

David Goldenheim and Daniel Litchfield are tenth graders at Greens Farms Academy, Greens Farms, CT 06436. Their Web site address is http://www.gfacademy.org. Goldenheim and Litchfield enjoy mathematics and computer science. Charles Dietrich, chdmath@gfacademy.org, teaches at Greens Farms Academy. In his thirty-three-year career as a mathematics teacher, he has also taught at Eton College in England and Westminster School in Connecticut.

- **Draw segment AM.**
- **Draw diagonal BD.**
- Let **G** be the point of intersection of $\overline{\mathbf{AM}}$ and $\overline{\mathbf{BD}}$.
- The foot of the altitude from **G** to $\overline{\mathbf{AB}}$ is $\mathbf{P_3}$, where $\mathbf{AP_3} = (1/3)\mathbf{AB}$. Note that when we write $\mathbf{P_n}$ for any positive integer *n,* we mean the point closest to **A** so that segment $\mathbf{AP_n}$ is the first part when segment **AB** is divided into *n* equal parts.
- **Draw segment $\mathbf{P_3D}$.**
- Let **H** be the point of intersection of $\overline{\mathbf{AM}}$ and $\overline{\mathbf{P_3D}}$.
- The foot of the altitude from **H** to $\overline{\mathbf{AB}}$ is $\mathbf{P_5}$, where $\mathbf{AP_5} = (1/5)\mathbf{AB}$.
- **Draw segment $\mathbf{P_5D}$.**
- Let **K** be the point of intersection of $\overline{\mathbf{AM}}$ and $\overline{\mathbf{P_5D}}$.
- The foot of the altitude from **K** to $\overline{\mathbf{AB}}$ is $\mathbf{P_7}$, where $\mathbf{AP_7} = (1/7)\mathbf{AB}$.

The algorithm being established can be repeated to find any unit fraction with an odd denominator.

To find a unit fraction with an even denominator, simply start this algorithm at $\mathbf{P_2}$, the midpoint of **AB** (**fig. 2**).

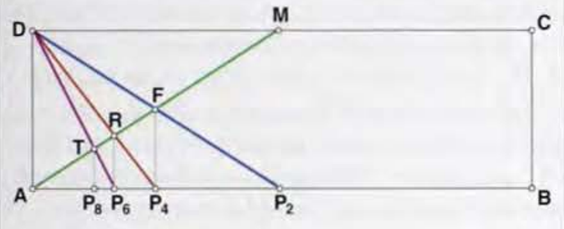

Fig. 2
$\mathbf{AP_n}$ are the unit fractions with even denominators of **AB.**

THE GEOMETRIC PROOF (DAVE)

Up to this point in the course we had focused equally on geometry and trigonometry. When Dan and I completed our pattern, we had to prove that it worked for any unit fraction. We decided to use geometry to prove the pattern. Within half an hour, we came up with a proof for specific points, such as the "one-third" point and the "one-fifth" point, but we could not come up with a proof for all points. Mr. Dietrich then showed us the principle of mathematical induction, its statement, its technique, and its power. With this information and his help, we were able to come up with a proof of the pattern for all unit fractions.

First, we establish the position of $\mathbf{P_3}$, such that $\mathbf{AP_3} = (1/3)\mathbf{AB}$. Clearly, $\triangle\mathbf{DGM} \sim \triangle\mathbf{BGA} \Rightarrow \mathbf{DM}/\mathbf{AB} = 1/2 \Rightarrow \mathbf{DM} = (1/2)\mathbf{AB}$. Construct the altitudes $\overline{\mathbf{GX}}$ to $\overline{\mathbf{AB}}$ and $\overline{\mathbf{GY}}$ to $\overline{\mathbf{DM}}$ (**fig. 3**). Since the lengths of the altitudes from corresponding vertices of similar triangles are in the same ratio as the lengths of corresponding sides, $\mathbf{YG}/\mathbf{XG} = 1/2$. Also, $\triangle\mathbf{MGY} \sim \triangle\mathbf{AGX} \Rightarrow \mathbf{YM}/\mathbf{AX} = \mathbf{YG}/\mathbf{XG} \Rightarrow \mathbf{YM}/\mathbf{AX} = 1/2$. In rectangle **AXYD,** opposite sides are congruent, so $\mathbf{AX} = \mathbf{DY}$. Therefore, $\mathbf{YM}/\mathbf{DY} = 1/2 \Rightarrow \mathbf{YM}/\mathbf{DY} + 1 = 1/2 + 1 \Rightarrow (\mathbf{YM} + \mathbf{DY})/\mathbf{DY} = 3/2$. Since the whole equals the sum of the parts, $\mathbf{DM}/\mathbf{DY} = 3/2 \Rightarrow \mathbf{DY}/\mathbf{DM} = 2/3 \Rightarrow \mathbf{DY} = (2/3)\mathbf{DM} = (2/3)(1/2)\mathbf{AB} = (1/3)\mathbf{AB} \Rightarrow \mathbf{AX} = (1/3)\mathbf{AB}$. Hence, $\mathbf{X} = \mathbf{P_3}$.

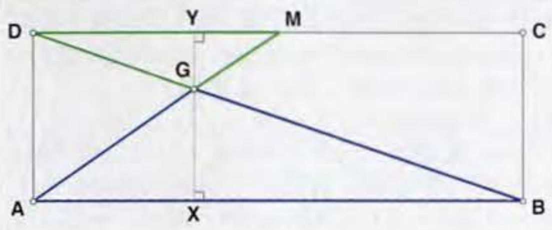

Fig. 3
Does $\mathbf{AX} = (1/3)\ \mathbf{AB}$?

Second, we give a proof, using the principle of mathematical induction, that allows us to obtain, in turn, division into fifths, sevenths, ninths, and so on. The principle of mathematical induction states the following:

> Let S_n be a statement involving the positive integer *n*. If (1) S_1 is true and (2) the truth of S_k implies the truth of S_{k+1} for every positive integer *k,* then S_k must be true for all positive integers *n*.

The statement S_n to be proved by the principle of mathematical induction is that the point $\mathbf{P_{2n+1}}$ found by the foregoing process satisfies $\mathbf{AP_{2n+1}} = [1/(2n+1)]\mathbf{AB}$ and so is a point of division for a unit fraction with an odd denominator.

The statement S_1 is simply that $\mathbf{AP_3} = (1/3)\mathbf{AB}$, which was previously proved. Assume that S_k is valid where *k is* some positive integer, so that

$$\mathbf{AP_{2k+1}} = \frac{1}{2k+1} \cdot \mathbf{AB}.$$

In **figure 4,** let $\mathbf{Z} = \mathbf{P_k}$.

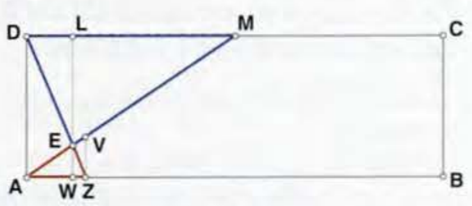

Fig. 4
Generalizing the pattern begun in **figure 3**

We next prove that S_{k+1} is valid. We know that $\triangle\mathbf{AZE} \sim \triangle\mathbf{MDE}$. Therefore,

$$\frac{\mathbf{AZ}}{\mathbf{DM}} = \frac{1}{2k+1} \cdot \frac{\mathbf{AB}}{\frac{1}{2}\mathbf{AB}} = \frac{2}{2k+1}.$$

Construct the altitudes $\overline{EL}$ to $\overline{DM}$ and $\overline{EW}$ to $\overline{AZ}$. Since the lengths of altitudes from corresponding vertices of similar triangles are in the same ratio as the lengths of corresponding sides, $WE/EL = AZ/DM = 2/(2k+1)$. By AA, $\triangle AEW \sim \triangle MEL \Rightarrow AW/ML = WE/EL \Rightarrow AW/ML = 2(2k+1)$. Opposite sides of rectangle **AWLD** are congruent, so $DL/ML = AW/ML = 2/(2k+1) \Rightarrow ML/DL = (2k+1)/2$. Since the whole equals the sum of the parts, $DL/DM = DL/(DL + LM)$. Dividing the numerator and denominator by **DL** yields $DL/(DL + LM) = 1/(1 + LM/DL) = 1/[1 + (2k + 1)/2] = 2/[2 + (2k + 1)] = 2/(2k + 3)$.

Hence $DL/DM = 2/(2k+3) \Rightarrow DL = [2/(2k+3)]DM = [2/(2k+3)][(1/2)AB] = [1/(2k+3)]AB$. Consequently, $AW = (1/[2(k+1)+1])\ AB \Rightarrow AW = AP_{2k+3} \Rightarrow S_{k+1}$ is valid.

Therefore, by the principle of mathematical induction, the proposition S_n is valid for all natural numbers. This result proves our construction for a unit fraction with any odd denominator.

It is important to verify that this process works for unit fractions with even denominators. We start at the midpoint P_2 of $\overline{AB}$, and we wish to generate the point **W** where $AW = (1/4)AB$ **(fig. 5).**

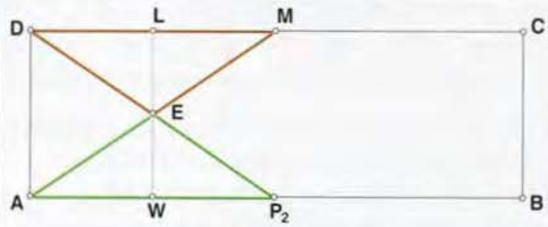

Fig. 5
Does the process work for the unit fractions with even denominators?

- On any segment **AB** construct any rectangle **ABCD.**
- Locate midpoints **M** of $\overline{DC}$ and P_2 of $\overline{AB}$.
- Draw segments **AM** and P_2D.
- Let **E** be the point of intersection of $\overline{AM}$ and $\overline{P_2D}$.
- The foot of the altitude from **E** to $\overline{AB}$ is **W.**

The proof that $W = P_4$ is similar to the arguments given previously.

A proof using the principle of mathematical induction can analogously be used to establish that $AP_n = (1/n)AB$ for even n. Consequently, the GLaD construction is hereby entirely proved geometrically.

THE ALGEBRAIC PROOF (DAN)

The algebraic proof, although fairly straightforward, was, I thought, the most interesting because it combined what we had learned the previous year with this new material. We continued this proof through the "one-ninth" point. To make sure that we had found a proof for all unit fractions, we tried using the principle of mathematical induction. After a few mishaps, we got the completed proof.

First, we wish to show that P_3 exists, where $P_3 = (1/3)\overline{AB}$. Let the length of $\overline{AB}$ be a and the length of $\overline{BC}$ be b and then use coordinates as given in **figure 6.** The equation of $\overleftrightarrow{AM}$ is $y = (2b/a)x$. The equation of $\overleftrightarrow{BD}$ is $y = (-b/a)x + b$. At point **G,** therefore, $(2b/a)x = (-b/a)x + b \Rightarrow x = a/3$. Since **X** is the foot of the altitude from **G** to $\overline{AB}$, $AX = (1/3)AB$. Therefore, $X = P_3$.

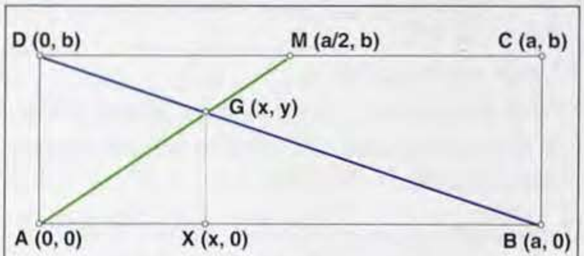

Fig. 6
Generalizing the pattern begun in figure 5

Second, we give a proof using the principle of mathematical induction that allows us to obtain, in turn, division into fifths, sevenths, ninths, and so on. The proposition to be proved by using the principle of mathematical induction is that P_{2n+1} obtained by the process described previously satisfies $AP_{2n+1} = [1/(2n + 1)]AB$ and so is a point of division for a unit fraction with an odd denominator. We have already proved that the process yields P_3 as desired. As an induction hypothesis, suppose that it yields P_{2k+1} such that $P_{2k+1} = Z$ has coordinates $(a/(2k + 1), 0)$. Let **W** be constructed as in **figure 7.** Recall that the equation of $\overleftrightarrow{AM}$ is $y = (2b/a)x$. The equation of $\overleftrightarrow{DZ}$ is $y = [-(2k + 1)b/a]x + b$. At point **E,** $(2b/a)x = [-(2k + 1)b/a]x + b \Rightarrow (b/a)[2 + (2k + 1)]x = b \Rightarrow x = a/(2k + 3) = a/[2(k + 1) + 1]$.

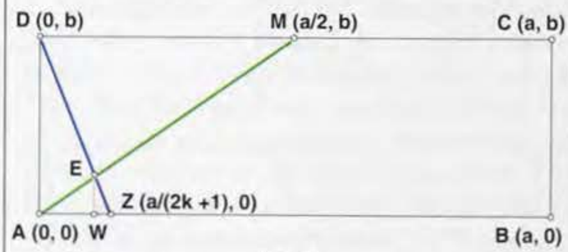

Fig. 7

$$AP_{2n+1} = \left[\frac{1}{2(2n+1)}\right] AB$$

Since **W** is the foot of the altitude from **E** to $\overline{AB}$, the coordinates of **W** are $(a/[2(k + 1) + 1], 0)$. Hence, $AW = 1/[2(k + 1) + 1]AB$ so that $W = P_{2k+3}$ satisfies $AP_{2k+3} = [1/(2k + 3)]AB$.

Thus by the principle of mathematical induction, we confirm our construction for a unit fraction with any odd denominator.

The proof for unit fractions with even denominators can be done analogously. Again, it is left as an exercise for the reader.

Consequently, the GLaD construction is hereby entirely proved algebraically. The challenge to divide any line segment into a regular partition of any number of parts has been met!

THE FIBONACCI SEQUENCE APPEARS! (DAVE)

When Dan and I began to figure out the pattern for dividing the line into unit fractions, we had some unsuccessful attempts. Toward the end of our work, I went back and decided to play with one of the first patterns, which used a reflection. I started that pattern and found that the reciprocals of the first three points were the beginning of the Fibonacci sequence. As I continued the pattern, I discovered that all the points in that pattern were reciprocals of numbers in the Fibonacci sequence. Mr. Dietrich was impressed—indeed, ecstatic—when I showed him what one of our first patterns turned out to be.

Leonardo Fibonacci, also known as Leonardo of Pisa (1170?–1240?), was an Italian mathematician. He is most famous for this particular sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... Each term of this sequence is called a *Fibonacci number.* After the second term, each term is the sum of the two numbers preceding it in the sequence. In general, a Fibonacci number would be $F_{n+2} = F_{n+1} + F_n$ for every natural number n. We are interested in the same sequence, but without the initial 1. Our sequence of reciprocals, therefore, would be 1/1, 1/2, 1/3, 1/5, 1/8, 1/13, 1/21, 1/34, ...

The construction

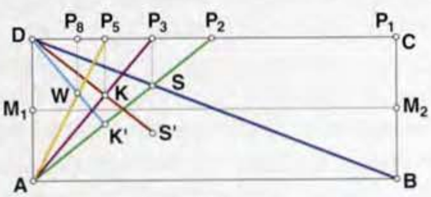

Fig. 8
The repetition of this pattern produces reciprocals of the Fibonacci numbers.

- On any line segment **AB** construct any rectangle **ABCD (fig. 8).**
- Find midpoints P_2 of $\overline{CD}$, M_1 of $\overline{AD}$, and M_2 of $\overline{BC}$.
- Draw segment M_1M_2.
- **Draw segment AP_2.**
- **Draw diagonal BD.**
- Let **S** be the point of intersection of $\overline{AP_2}$ and $\overline{BD}$.
- The foot of the altitude from **S** to $\overline{CD}$ is P_3, where $DP_3 = (1/3)DC$ (see the foregoing geometric proof).
- Let **S'** be the point of reflection of **S** over $\overline{M_1M_2}$.
- **Draw $\overline{S'D}$.**
- **Draw $\overline{AP_3}$.**
- Let **K** be the point of intersection of $\overline{AP_3}$ and $\overline{S'D}$.
- The foot of the altitude from **K** to $\overline{CD}$ is P_5, where $DP_5 = (1/5)DC$, which is shown subsequently.
- Let **K'** be the point of reflection of **K** over $\overline{M_1M_2}$.
- **Draw $\overline{K'D}$.**
- **Draw $\overline{AP_5}$.**
- Let **W** be the point of intersection of $\overline{AP_5}$ and $\overline{K'D}$.
- The foot of the altitude from **W** to $\overline{CD}$ is P_8, where $DP_8 = (1/8)DC$.

The algorithm to be established can be repeated to find the sequence of reciprocals of all numbers contained in the Fibonacci sequence.

Let **B** be the reflection of P_1 over the perpendicular bisector $\overline{M_1M_2}$ of $\overline{DA}$ **(fig. 9).** Draw diagonal **BD** and segment AP_2. Let **S** be the point of intersection of $\overline{BD}$ and $\overline{AP_2}$. The foot of the altitude from **S** to $\overline{P_1D}$ would be P_3, as previously proved geometrically and algebraically.

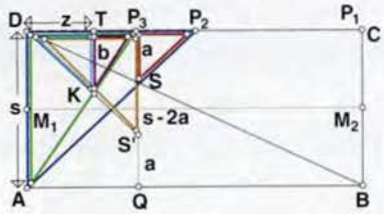

Fig. 9
Points in the pattern are reciprocals of the Fibonacci numbers.

Given points P_2 and P_3, we wish to show that **T** is the point P_5 with $DP_5 = (1/5)DC$. Let $AB = 1$, $SP_3 = a$, $TK = b$, $DA = s$, and $DT = z$. Since **S'** is the reflection of **S** over the perpendicular bisector $\overline{M_1M_2}$ of $\overline{DA}$, then $S'Q = a$.

$$\triangle P_2P_3S \sim \triangle P_2DA \Rightarrow P_2P_3/P_2D = P_3S/DA \Rightarrow (1/2 - 1/3)/(1/2) = a/s \Rightarrow a = (1/3)s$$

$$\triangle P_3TK \sim \triangle P_3DA \Rightarrow P_3T/P_3D = TK/DA \Rightarrow (1/3 - z)/(1/3) = b/s \Rightarrow b = 3s(1/3 - z)$$

$$\triangle DKT \sim \triangle DS'P_3 \Rightarrow DT/DP_3 = TK/S'P_3 \Rightarrow z = (1/3)(b)/(s - a)$$

Therefore by substitution, $z = (1/3)[3s(1/3 - z)]/[s - (1/3)s] = (3/2)(1/3 - z) = 1/2 - (3/2)z \Rightarrow (5/2)z = 1/2 \Rightarrow z = 1/5$. So **T** is proved to be P_5.

To obtain the general Fibonacci result, consider **figure 10,** where $AD = s$, $DT = z$, $DF = y$, $DE = x$, $FS = a$, $FS' = c$, and $TK = b$. Since **S'** is the reflection of **S** over the perpendicular bisector $\overline{M_1M_2}$ of $\overline{AD}$, then $S'Q = a$.

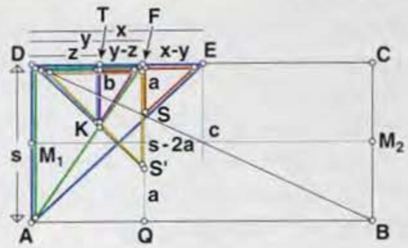

Fig. 10
Obtaining the general Fibonacci result

By AA, $\triangle EFS \sim \triangle EDA \Rightarrow FS/DA = EF/ED \Rightarrow a/s = (x-y)/x \Rightarrow a = s(x-y)/x = s - sy/x$. Hence, $c = s - a = sy/x \Rightarrow c/y = s/x$.

By AA, $\triangle DTK \sim \triangle DFS' \Rightarrow b/z = c/y = s/x \Rightarrow b/s = z/x$.

By AA, $\triangle FTK \sim \triangle FDA \Rightarrow b/s = (y-z)/y = 1 - (z/y)$. Hence, $z/x = 1 - (z/y) \Rightarrow 1/x = 1/z - 1/y \Rightarrow 1/x + 1/y = 1/z$.

In particular, if F_n and F_{n+1} are consecutive terms of the Fibonacci sequence and if $x = 1/F_n$ and $y = 1/F_{n+1}$, then $z = 1/F_{n+2}$ and the result concerning reciprocals of the Fibonacci sequence is established.

CONCLUSION

Dan and Dave. In retrospect, we can see how important The Geometer's Sketchpad was in the formation of this construction. It enabled us to work with ideas, see patterns, and test our theories. Once we were convinced of our discoveries, proving them was the icing on the cake. Don't you agree, Mr. Dietrich?

Charles Dietrich. Yes. For me the thrill of this adventure has not receded over time. I am staggered still by the amount of different mathematics at the high school level that can be used in this problem, I am amazed at the simplicity of the GLaD construction, and I am agog that I was involved with creating something in mathematics. It has been a mathematician's dream. My hope is that this construction will appeal to many teachers and exceptional students whose interests include geometry, algebra, number theory, technology, inductive reasoning and proofs, and assorted other topics.

We wish to thank the three referees, particularly James Bristol of Shaker Heights, Ohio, who encouraged us, inspired us, and provided us with some useful and helpful ideas. They have greatly enhanced our article. Above all, we ask anyone who does further work on our ideas to share his or her conclusions with us, please!