Parabolic Properties from Pieces of String

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Illuminating Illustration: Parabolic Properties from Pieces of String

Depending on the context, the word *parabola* could refer to either the graph of a quadratic polynomial or (figure 1) the set of all points which are the same distance from a given point F (called the *focus*) and a given line (called the *directrix* and shown in red). To motivate the study of parabolas in either context, textbooks often mention that parabolic reflectors can direct sound and light in devices such as telescopes, satellite dishes, lighting fixtures, and microphones. In figure 1, let L (for light) be an arbitrary point above R, and imagine a ray of light traveling downward along $\overline{LR}$, a line segment parallel to the parabola's axis of symmetry and perpendicular to the directrix. After hitting the parabola at R, the light reflects as if the blue

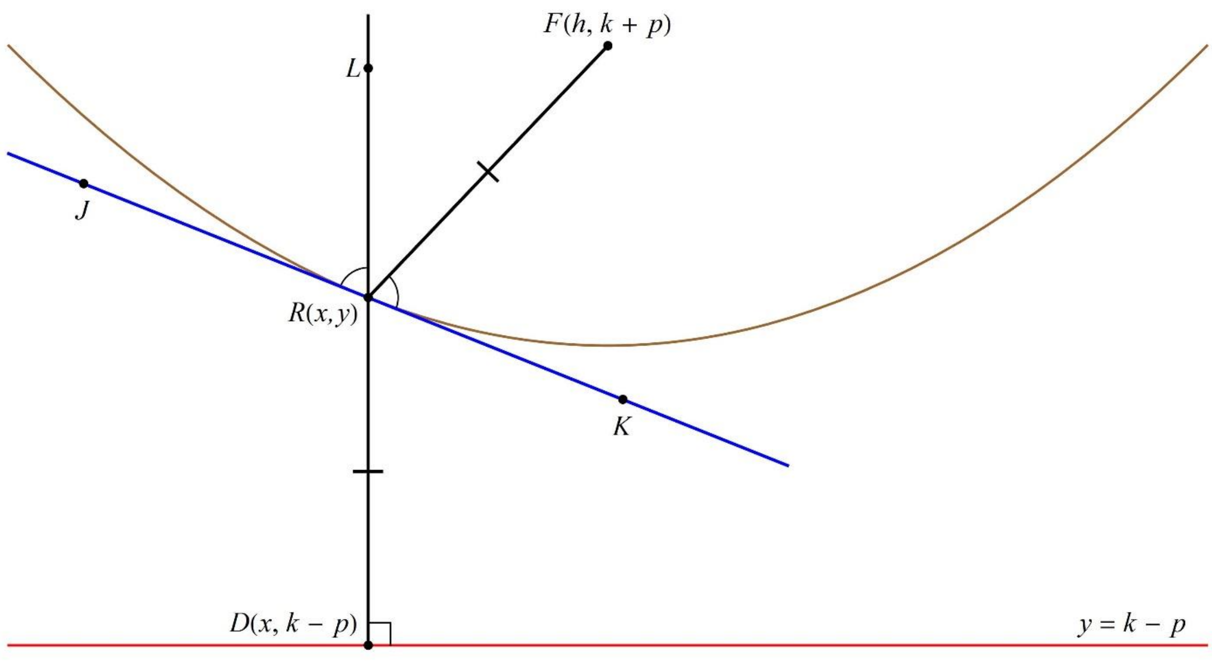

Figure 1. A point *R* on a parabola satisfies *RF* = *RD*, where *F* is the focus and *D* is the nearest point to *R* on the directrix. Also, the reflective property of parabolas states that a beam of light that is perpendicular to the directrix will reflect at any point *R* to the focus *F*.

tangent line at *R* was a mirror. The reflective property of parabolas states that, no matter where the incoming ray of light hits the parabola, the reflected light passes through the focus *F*.

The usual proof of the reflective property begins with the focus-directrix definition of a parabola. Suppose that *F* has coordinates $(h, k + p)$, the directrix has equation $y = k - p$, and *D* is the point on the directrix closest to *R*. Using $RF = RD$ and the distance formula, we obtain

$$\sqrt{(x-h)^2+(y-k-p)^2} = y - k + p,$$

which simplifies to the standard form $(x - h)^2 = 4p(y - k)$ of a parabola. To prove the reflective property, one can find dy/dx (and thus the slope of the blue tangent line) and show that the angles $\angle JRL$ and $\angle KRF$ are congruent.

In this article, we present a different proof of the reflective property that does not require calculus. We reverse the usual logic and show that the graph of a quadratic polynomial has a focus and directrix, and the reflective property will be a natural corollary of this geometric argument. Along the way, we will also find the tangent line of a parabola without using calculus. We begin our explorations with a third way of creating parabolas: string art.

String art

When I was a child, I enjoyed playing with string art, which can be created by drawing line segments, which we will call *strings*, with endpoints chosen from equally spaced points along two given line segments. Figure 2 shows string art from the line segments that connect *B*(8, 0) to *A*(0, 8) and to *C*(16, 8). Seven strings are also shown, connecting (1, 7) to (9, 1), (2, 6) to (10, 2), and so on. Evidently, the strings trace some kind of curve.

Most mathematical studies of string art (formally called *envelopes*) rely on differential equations. However, since string art is simple enough for a young child to construct, we will instead find the apparent curve in figure 2 using simpler mathematical tools. The colored points indicate the points on the strings with the largest y-coordinate at $x = 2, 4, 6, \ldots, 14$. For example, the dashed vertical line $x = 4$ intersects the red, brown, orange, and green strings; of these, the brown string has the highest point of intersection. The brown string connecting (2, 6) and (10, 2) has equation $y = -0.5x + 7$, and so the y-coordinate of the brown point is $y = -0.5(4) + 7 = 5$.

Assuming the string art curve is the graph of a quadratic polynomial, it would make sense that the graph passes through $A(0, 8)$, $C(16, 8)$, and the (presumed) vertex $V(8, 4)$. The reader is invited to show that quadratic polynomial whose graph contains A, C, and V is $y = x^2/16 - x + 8$ and, furthermore, that the graph indeed passes through all of colored points in figure 2.

To formalize this heuristic argument, consider line segments $\overline{AB}$ and $\overline{BC}$ with endpoints $A(0, t)$, $B(t, 0)$, and $C(2t, t)$, with $t > 0$ (figure 3). Also shown in red is "string s," where

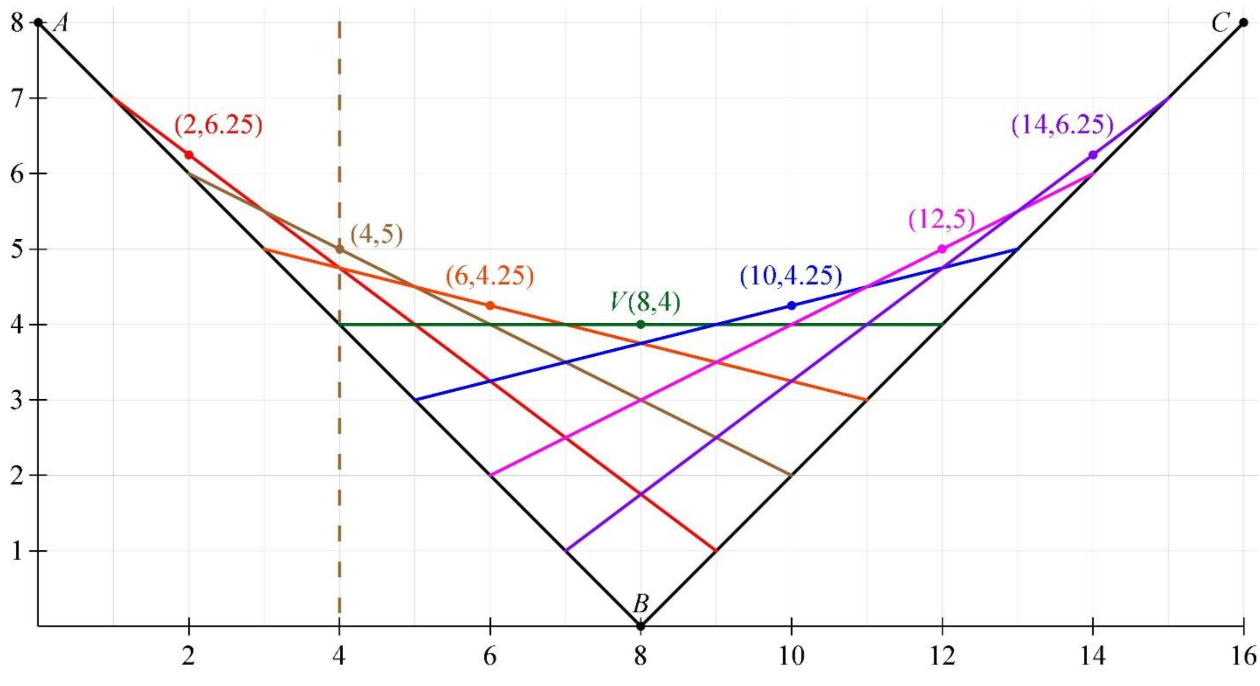

Figure 2. An example of string art. The colored points indicate which string has the largest y-coordinate at $x = 2, 4, 6, \ldots, 14$. Direct calculations confirm that these points also lie on the graph of $y = x^2/16 - x + 8$.

s is the x-coordinate of the string's left endpoint P. The reader is invited to use the endpoints $P(s, t - s)$ and $Q(s + t, s)$ to show that string s has equation $y = (2s / t - 1)x + t - 2s^2 / t$. For example, if $s = 2$ and $t = 8$, we obtain $y = -0.5x + 7$, which was the equation of the brown string in figure 2.

To find the string art curve, we now find the string s that maximizes $y = (2s / t - 1)x + t - 2s^2 / t$, where x and t are fixed. We invite the reader to solve $dy/ds = 0$ (or, for a calculus-free proof, to find the vertex of this quadratic polynomial in s) and show that the optimal value of s is $x / 2$. Substituting back into the equation for y gives the value of this maximal y-coordinate:

$$y = (2(x / 2) / t - 1)x + t - 2(x /2)^2 / t = x^2 / (2t) - x + t.$$

Summarizing, we have shown that the string art curve is the graph of a quadratic polynomial. (We can recover the quadratic polynomial for figure 2 by setting t equal to 8.)

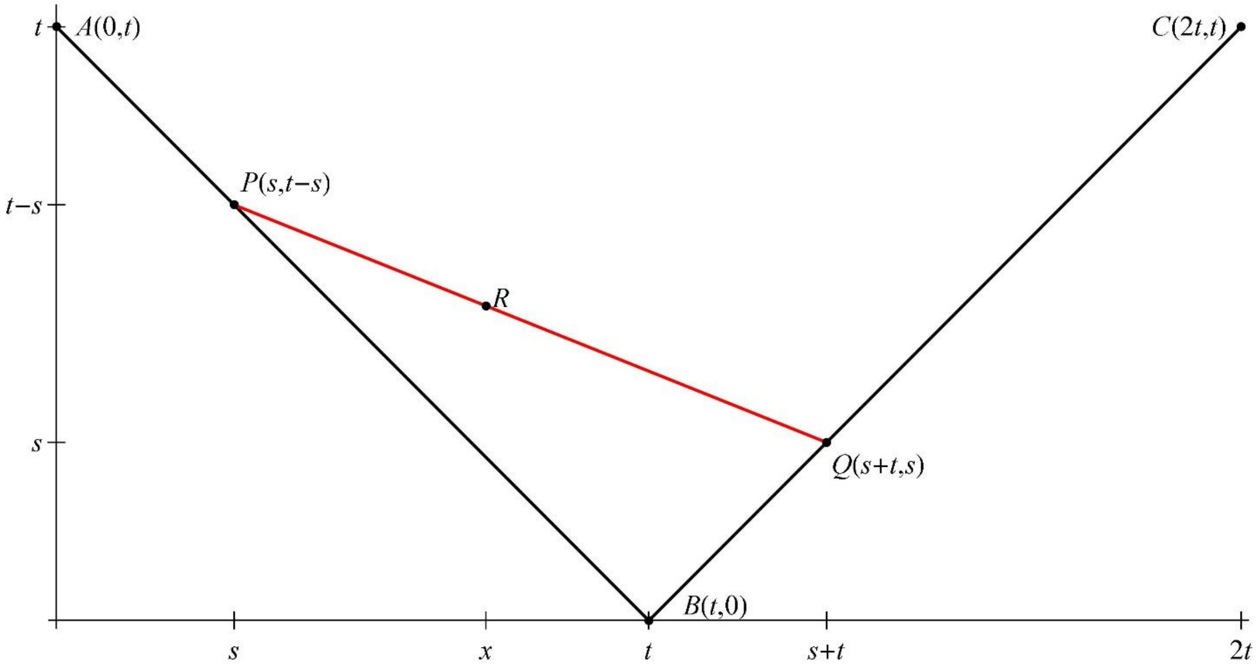

Figure 3. The coordinates of R on $\overline{PQ}$, denoted as "string s," are $(x, (2s / t - 1)x + t - 2s^2 / t)$, where x, s, and t are the x-coordinates of R, P, and B, respectively. For a fixed value of x, the y-coordinate of R has a maximum value of $x^2 / (2t) - x + t$ when $s = x / 2$.

Although figure 3 depicts the case $0 < s < t$ (that is, P is between A and B and Q is between B and C), the above derivation applies even if $s \leq 0$ or $s \geq t$. Therefore, the graph of this quadratic polynomial will continue to be traced if strings connecting equally-spaced points past the endpoints A and C are drawn and extended, as shown in figure 4.

Interestingly, since $s = x / 2$ is a one-to-one function, we have shown that string $s = x / 2$ is the *only* string that passes through a given point (x, y) on the graph of this quadratic polynomial. In other words, we have shown that string s is tangent to the curve at $x = 2s$. We invite the reader to use calculus to find the tangent line of $y = x^2 / (2t) - x + t$ at $x = 2s$ and confirm that the points $P(s, t - s)$ and $Q(s + t, s)$ both lie on this tangent line.

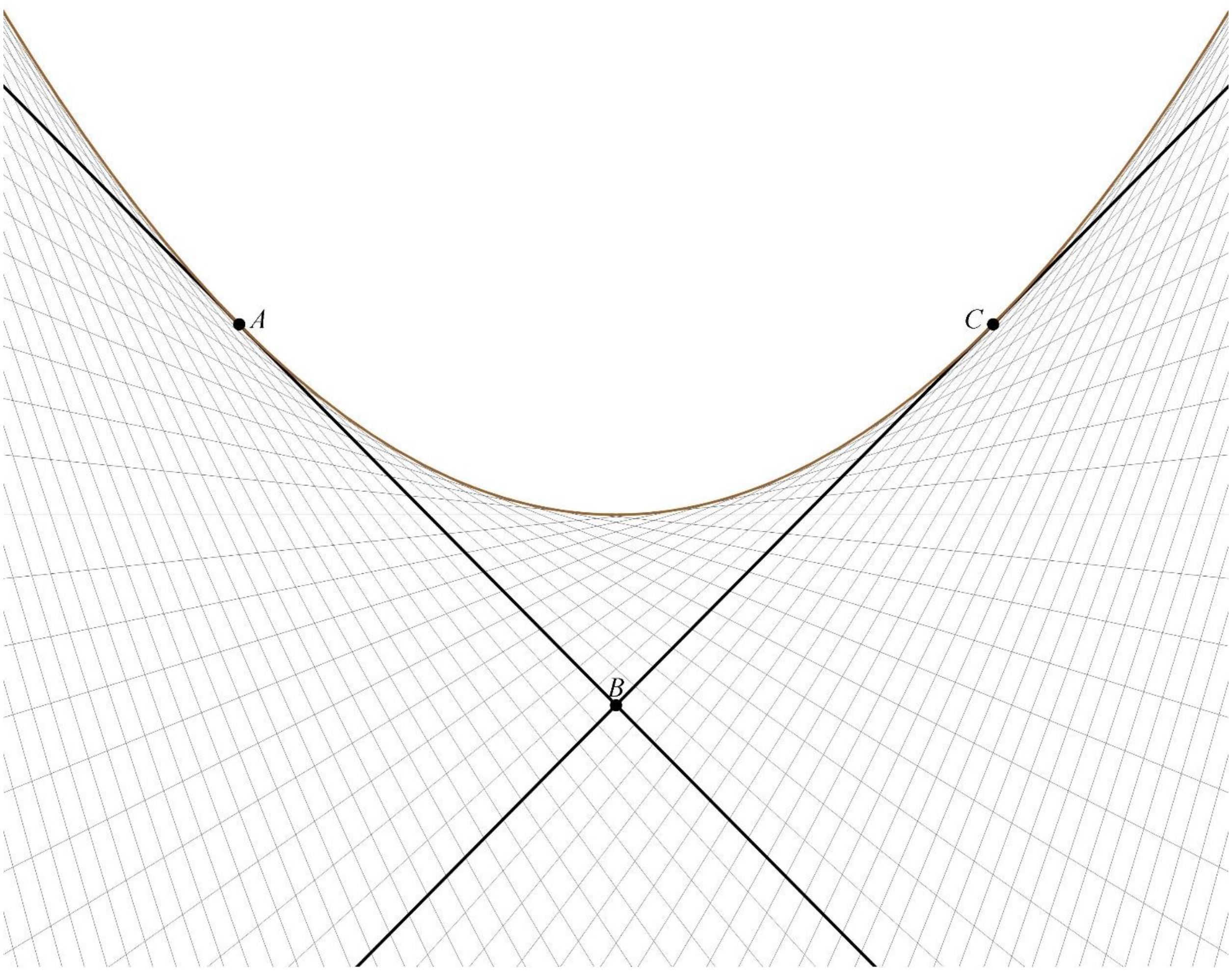

Figure 4. Extended string art traces the full graph of the quadratic polynomial.

Focus-directrix property

Our study of string art has put us in position to show that the graph of $y = x^2 / (2t) - x + t$ has a focus and directrix. Figure 5(a) shows this graph and the line segments $\overline{AB}$ and $\overline{BC}$. An arbitrary point *R* on the graph, with *x*-coordinate 2*s*, is also shown. (Although figure 5(a) depicts *R* as between *A* and *C*, the argument that follows still works if *R* is either to the left of *A* or to the right of *C*.) As discussed earlier, string *s*, with endpoints $P(s, t - s)$ and $Q(s + t, s)$, is tangent to the graph at *R*. Figure 5(a) also shows $\overline{PF}$ and $\overline{FQ}$, where $F(t, t)$ is the midpoint of $\overline{AC}$. (The reader may guess why *F* was chosen as the name of this point.) Since right triangles ΔPXF and ΔFYQ are congruent by the SAS postulate, angles $\angle PFX$ and $\angle QFY$ are complementary. Therefore, $\overline{PF}$ and $\overline{FQ}$ are both congruent and perpendicular.

Define the point *D* so that quadrilateral *DPFQ* is a square, as shown in figure 5(b). (The reader may also guess why *D* was chosen as the name of this point.) A quick calculation shows that the coordinates of *D* are $(2s, 0)$, and so *D* lies on the *x*-axis directly below *R* for any value of *s*. In other words, *RD* is the distance from *R* to the *x*-axis.

Figure 5(b) also shows triangles ΔDRM and ΔFRM, where *M* is the intersection of $\overline{PQ}$ and $\overline{DF}$. Since the diagonals of a square are perpendicular bisectors of each other, we see that $DM = FM$ and that $\angle FMR$ and $\angle DMR$ are right angles. Furthermore, triangles ΔDRM and ΔFRM are congruent by the SAS postulate, and so corresponding line segments $\overline{RD}$ and $\overline{RF}$ are congruent. We conclude that, for any point *R* on the graph of $y = x^2 / (2t) - x + t$, the distance from *R* to *F* (the focus of the parabola) equals the distance from *R* to the *x*-axis (the directrix).

We invite the reader to reverse this geometric argument and confirm with conic sections that the parabola with focus (t, t) and directrix $y = 0$ has equation $y = x^2 / (2t) - x + t$.

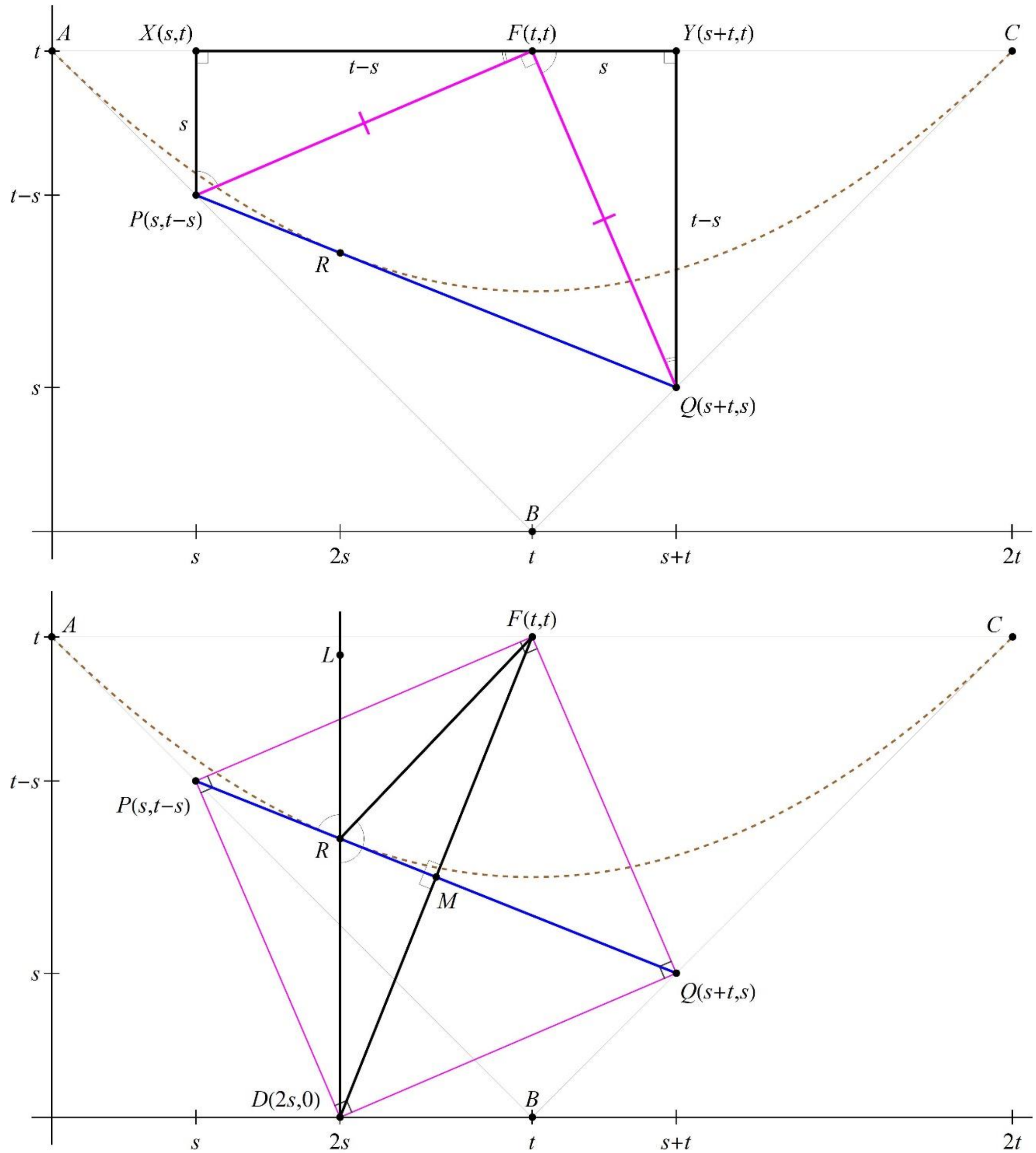

Figure 5. (a) Connecting the midpoint *F* of $\overline{AC}$ to the endpoints of any string *s* forms congruent and perpendicular line segments $\overline{PF}$ and $\overline{FQ}$. (b) Square *DPFQ* and the congruence of right triangles Δ*DRM* and Δ*FRM* prove that the graph of a quadratic polynomial is satisfies the conic-section definition of a parabola with focus *F* and directrix the *x*-axis. The figure also shows that ∠*LRP* and ∠*FRM* are congruent, thus proving the reflective property of parabolas.

Reflective property

The reflective property of parabolas follows immediately from figure 5(b). Vertical line segment $\overline{DL}$ intersects $\overline{PQ}$ at R, so that $\angle LRP$ and $\angle MRD$ are vertical angles. Also, $\angle MRD$ and $\angle FRM$ are the corresponding parts of congruent triangles. Therefore, $\angle LRP$ and $\angle FRM$ are congruent, proving the reflective property for the parabola $y = x^2 / (2t) - x + t$. Furthermore, since t is an arbitrary positive number, rotations and translations (figure 6) can be used to prove the reflective property for all parabolas, whether opening upward, downward, or at an angle.

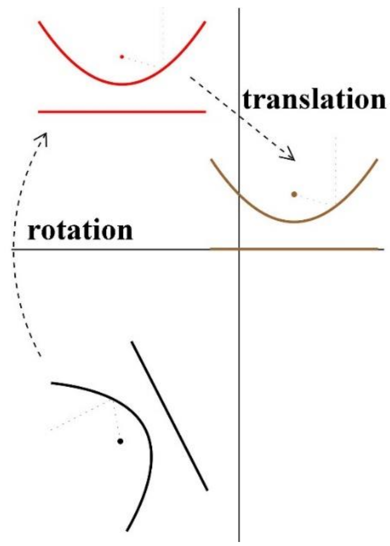

Figure 6. Any parabola (black) may be transformed into the graph (brown) of $y = x^2 / (2t) - x + t$ by applying a rotation and/or translation, thus proving the reflective property for all parabolas.

FURTHER READING

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